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MASS TRANSPORT AND THE FORCE OF A BEAM OF TWO-DIMENSIONAL PERIODIC INTERNAL WAVES†

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The steady flow generated by an arbitrary field of monochromatic internal waves in a viscous continuously stratified liquid is calculated in the first order of perturbation theory. The streamline pattern is calculated for a beam of harmonic waves excited by a point dipole. The steady force, which a beam incident on rigid plane surface exerts, is also calculated. © 2001 Elsevier Science Ltd. All rights reserved.

A calculation of the mass transport by wave motions (Stokes drift) in the case of surface waves [1, 2] agrees with experiments [3]; when calculating the wave drift at the interfaces of uniform homogeneous liquids, the effect of the boundary layers that arise is also taken into account [4]. Stokes drift in the theory of internal waves in a continuously stratified liquid, the equations of motion of which [5] differ considerably from the equations of the theory of surface waves, has been investigated to a lesser extent. The development of the theory of slightly non-linear internal waves in a viscous medium, which satisfy the boundary conditions exactly, enables one to calculate not only wave mass transport but also the force of the waves on a reflecting surface, which is important for problems of the dynamics of the interaction of the atmosphere and the underlying surface, and estimates of the action of oceanic internal waves on large-scale structures.

The purpose of this paper is to calculate the average velocities, the distortions of the stratification and the force on an obstacle due to a field of two-dimensional monochromatic internal waves in a viscous continuously stratified medium.

1. THE EQUATIONS OF MOTION AND THE BOUNDARY CONDITIONS

The system of equations of the two-dimensional motion of a viscous incompressible continuously stratified liquid in a system of coordinates (x, y, z) has the form [6].

$$\begin{aligned}
&\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial P}{\partial x}+\nu\rho\Delta v_x+\\
&+2\nu\frac{\partial\rho}{\partial x}\frac{\partial v_x}{\partial x}+\nu\frac{\partial\rho}{\partial z}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)+f^x\\
&\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial P}{\partial z}+\nu\rho\Delta v_z+\\
&+\nu\frac{\partial\rho}{\partial x}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)+2\nu\frac{\partial\rho}{\partial z}\frac{\partial v_z}{\partial z}-\rho g+f^z\\
&\frac{\partial\rho}{\partial t}+v_x\frac{\partial\rho}{\partial x}+v_z\frac{\partial\rho}{\partial z}=0,\quad \frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}=0;\quad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}
\end{aligned}\tag{1.1}$$

where ρ, P, (v_x, v_z) are the total density, pressure and velocity, ν is the kinematic viscosity, which is assumed to be constant, and (f^x, f^z) are the force sources which give rise to the motion of the liquid; the z axis is directed in the opposite direction to the acceleration due to gravity g.

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We will consider the motions in a medium with an arbitrary continuous density distribution $\rho_0(z)$ and a corresponding hydrostatic pressure profile

$$P_0(z) \equiv P_0(z_0) - g\int_{z_0}^{z}\rho_0(\zeta)d\zeta$$

The boundary conditions are the no-slip conditions for the velocity on the solid surfaces S in the liquid: $v_x\, s = v_z\, s = 0$, and also the decay of all the perturbations at infinity.

Taking into account the two-dimensional nature of the problem and the incompressibility of the liquid, in the calculations below we will use the stream function Ψ, related to the velocity as follows:

$$v_x = \partial\Psi/\partial z, \quad v_z = -\partial\Psi/\partial x \tag{1.2}$$

Introducing the density and pressure perturbations

$$P \to P_0(z) + P, \qquad \rho \to \rho_0(z) + \rho \tag{1.3}$$

and substituting expressions (1.2) and (1.3) into Eq. (1.1), we obtain the system of equations

$$\begin{aligned}
&(\rho_0+\rho)(\Psi_{tz}+\Psi_z\Psi_{xz}-\Psi_x\Psi_{zz}) = -P_x + \nu(\rho_0+\rho)\Delta\Psi_z + \\
&+\ 2\nu\rho_x\Psi_{xz} + \nu(\rho_0'+\rho_z)(\Psi_{zz}-\Psi_{xx}) + f^x \\
&(\rho_0+\rho)(-\Psi_{tx}-\Psi_z\Psi_{xx}+\Psi_x\Psi_{xz}) = -P_z - \nu(\rho_0+\rho)\Delta\Psi_x + \\
&+\ \nu\rho_x(\Psi_{zz}-\Psi_{xx}) - 2\nu(\rho_0'+\rho_z)\Psi_{xz} - \rho g + f^z \\
&\rho_t + \rho_x\Psi_z - (\rho_0'+\rho_z)\Psi_x = 0
\end{aligned} \tag{1.4}$$

Here and henceforth the subscripts t, x and z denote differentiation with respect to the corresponding variable.

2. STOKES FLOW IN AN INTERNAL-WAVE FIELD

We will calculate the flow generated due to non-linear effects by an arbitrary field of monochromatic internal waves with frequency ω, the stream function of which is described by the relation

$$\Psi_1 = \mathrm{Re}[\psi(x,z)e^{-i\omega t}] = \frac{1}{2}[\psi(x,z)e^{-i\omega t} + \Psi^*(x,z)e^{i\omega t}] \tag{2.1}$$

where the asterisk denotes complex conjugation. The initial field (2.1) generates a steady (Stokes) flow and waves of higher harmonics. The field Ψ_1 satisfies the following linearized systems of equations from (1.4)

$$\begin{aligned}
&\rho_0\Psi_{1tz} = -P_{1x} + \nu\rho_0\Delta\Psi_{1z} + \nu\rho_0'(\Psi_{1zz}-\Psi_{1xx}) + f^x \\
&\rho_0\Psi_{1tx} = P_{1z} + \nu\rho_0\Delta\Psi_{1x} + 2\nu\rho_0'\Psi_{1xz} + \rho_1 g - f^z \\
&\rho_{1t} - \rho_0'\Psi_{1x} = 0
\end{aligned} \tag{2.2}$$

We will seek a solution of the non-linear system of equations (1.4) in the form of expansions in harmonics of the frequency ω for the stream function

$$\Psi(x,z,t) = \Psi_s(x,z) + \sum_{n=1}^{\infty}[\Psi_n^+(x,z)e^{in\omega t} + \Psi_n^-(x,z)e^{-in\omega t}] \tag{2.3}$$

and similar expansions for the pressure and density. Here and henceforth terms with the subscripts describe Stokes flow. In the zeroth approximation, non-linear corrections to the initial field with frequency ω can be neglected, and we can assume

$$\Psi_1^+ = \tfrac{1}{2}\psi^*(x,z), \quad \Psi_1^- = \tfrac{1}{2}\psi(x,z) \tag{2.4}$$

It follows from Eqs (2.2) that in the part of space free from sources the function ψ satisfies the equation of the internal waves

$$\Delta\psi-\frac{N^2}{\omega^2}\psi_{xx}-\frac{i\nu}{\omega}\Delta^2\psi+\frac{\rho_0'}{\rho_0}\psi_z-\frac{2i\nu\rho_0'}{\omega\rho_0}\Delta\psi_z-\frac{i\nu\rho_0''}{\omega\rho_0}(\psi_{zz}-\psi_{xx})=0 \tag{2.5}$$

in deriving which we have not use the traditional Boussinesq approximation. Here $N^2(z)=g/\Lambda$ is the square of the buoyancy frequency and $\Lambda=[d\ln\rho_0(z)/dz]^{-1}$ is the scale of the stratification. Substituting (2.3) and similar expressions for the pressure and density into system (1.4) and retaining only quadratic terms in Ψ_1, we obtain, taking into account Eqs (2.2), the following system of equations in Ψ_s, P_s and ρ_s

$$\begin{aligned}
&(\rho_0+\rho_s)(\Psi_{sz}\Psi_{sxz}-\Psi_{sx}\Psi_{szz})+P_{sx}-\nu(\rho_0+\rho_s)\Delta\Psi_{sz}-\\
&-2\nu\rho_{sx}\Psi_{sxz}-\nu(\rho_0'+\rho_{sz})(\Psi_{szz}-\Psi_{sxx})=F^x\\
&(\rho_0+\rho_s)(-\Psi_{sz}\Psi_{sxx}+\Psi_{sx}\Psi_{sxz})+P_{sz}+\nu(\rho_0+\rho_s)\Delta\Psi_{sx}+\\
&+\ \nu\rho_{sx}(\Psi_{sxx}-\Psi_{szz})+2\nu(\rho_0'+\rho_{sz})\Psi_{sxz}+\rho_s g=F^z\\
&\rho_{sx}\Psi_{sz}-(\rho_0'+\rho_{sz})\Psi_{sx}=Q
\end{aligned}\tag{2.6}$$

where

$$\begin{aligned}
&F^x=\langle-\rho_1\Psi_{1tz}-\rho_0(\Psi_{1z}\Psi_{1xz}-\Psi_{1x}\Psi_{1zz})+\\
&+\nu[\rho_1\Delta\Psi_{1z}+2\rho_{1x}\Psi_{1xz}+\rho_{1z}(\Psi_{1zz}-\Psi_{1xx})]\rangle\\
&F^z=\langle-\rho_1\Psi_{1tx}-\rho_0(\Psi_{1x}\Psi_{1xz}-\Psi_{1z}\Psi_{1xx})-\\
&-\nu[\rho_1\Delta\Psi_{1x}+2\rho_{1z}\Psi_{1xz}+\rho_{1x}(\Psi_{1xx}-\Psi_{1zz})]\rangle\\
&Q=\langle\rho_{1z}\Psi_{1x}-\rho_{1x}\Psi_{1z}\rangle
\end{aligned}\tag{2.7}$$

The angular brackets denote averaging over the period $2\pi/\omega$.

Hence, in the approximation considered, the non-linear terms in system (1.1) are replaced by effective stationary force sources ($F^x(x,z)$ and $F^z(x,z)$) in the equations of motion and by a mass source with flow rate $Q(x,z)$ in the equation of continuity.

It follows from (2.1) and the last equation in (2.2) that

$$\rho_1=\frac{i\rho_0'}{2\omega}(\psi_x e^{-i\omega t}-\psi_x^* e^{i\omega t})$$

Substituting this expression and relation (2.1) into (2.7) we obtain

$$\begin{aligned}
&F^x=\frac{\rho_0}{4}\Big\{\psi_z\psi_{xz}^*-\psi_x\psi_{zz}^*-\frac{\psi_x\psi_z^*}{\Lambda}-\\
&-\frac{i\nu}{\omega\Lambda}[\psi_x\Delta\psi_z^*+\psi_{xz}(\psi_{zz}^*-3\psi_{xx}^*)]\Big\}+\text{c.c.}\\
&F^z=\frac{\rho_0}{4}\Big\{\psi_z\psi_{xx}^*-\psi_x\psi_{xz}^*+\frac{\psi_x\psi_x^*}{\Lambda}+\\
&+\frac{i\nu}{\omega\Lambda}[\psi_x\Delta\psi_x^*+\psi_{xx}(\psi_{xx}^*-\psi_{zz}^*)]\Big\}+\text{c.c.}\\
&Q=-\frac{i\rho_0'}{4\omega}\frac{\partial}{\partial x}(\psi_x\psi_z^*-\psi_z\psi_x^*)
\end{aligned}\tag{2.8}$$

where c.c. denotes the complex-conjugate quantity. It follows from the form of the expression for Q in (2.8) that such a source has zero total mass transport ($\iint Q(x,z)dxdz=0$).

Assuming that the velocities of the induced wave flow are small, system (2.6) can be linearized:

$$P_{sx} - \nu\rho_0\Delta\Psi_{sz} + \nu\rho_0'(\Psi_{sxx} - \Psi_{szz}) = F^x$$
$$P_{sz} + \nu\rho_0\Delta\Psi_{sx} + 2\nu\rho_0'\Psi_{sxz} + \rho_s g = F^z \quad (2.9)$$
$$-\rho_0'\Psi_{sx} = Q$$

Comparing the last equations in systems (2.8) and (2.9), we obtain the stream function of the Stokes flow generated by an arbitrary field of monochromatic internal waves,

$$\Psi_s(x,z) = \frac{i}{4\omega}[\psi_x(x,z)\psi_z^*(x,z) - \psi_z(x,z)\psi_x^*(x,z)] \quad (2.10)$$

It can be seen that if the field $\psi(x, z)$ satisfies the no-slip conditions, we have

$$\Psi_{sx}\big|_S = \Psi_{sz}\big|_S = 0$$

i.e. the Stokes flow also satisfies the boundary conditions exactly. Substituting the stream function of the isolated wave beam excited by a compact source [7] into expression (2.10), we obtain that the velocity of the induced flow vanishes in the ideal-liquid approximation ($\nu = 0$). In regions where several wave beams intersect, a Stokes drift also occurs when there is no viscosity.

Eliminating the pressure P_s from system (2.9) and using relations (2.8), we obtain the following equation for the density perturbation in Stokes flow

$$\frac{4g}{\rho_0}\rho_{sx} \approx (\psi_z\Delta\psi_x^* - \psi_x\Delta\psi_z^*) + \frac{2\psi_x\Delta\psi^*}{\Lambda} - \frac{i\nu}{\omega}\Delta^2\psi_x\psi_z^* + \text{c.c.} \quad (2.11)$$

Here we have omitted terms of the following orders of smallness in $1/\Lambda$.

The relative order of smallness of the terms in relation (2.11) depends on the position of the point of observation. Far from sources and reflecting surfaces, where the stream function of the initial field varies slowly, the principal term is the first one. The smallness of the other two terms follows from the condition of weak stratification and low viscosity ($\Lambda \gg \lambda \gg \delta_\nu$, where λ is the characteristic scale of the beam and $\delta_\nu = \sqrt{\nu/\omega}$ is the transverse scale of the internal boundary flow [5]), which is satisfied both under laboratory and natural conditions. In the region of sources and reflecting surfaces, where internal boundary flows exist in addition to waves [8], the principal terms are the first and third, since here the action of the Laplace operator Δ on the function is equivalent to multiplying it by the quantity ω/ν. In the intermediate region all three terms are comparable in value, when the characteristic spatial scale of the internal waves λ is of the order of the viscous wave scale $L_\nu = (\nu g)^{1/3}/N$ [7].

Integrating relations (2.11) taking Eqs (2.5) into account and retaining only the principal term, we obtain that, far from sources and bounding surfaces, the density perturbation in Stokes flow has the form

$$\rho_s(x,z) = \frac{\rho_0 N^2}{4g\omega^2}[\psi_z\psi_{xx}^* - \psi_x\psi_{xz}^* + \psi_z^*\psi_{xx} - \psi_x^*\psi_{xz}] \quad (2.12)$$

As an example, consider Stokes flow induced by a beam of internal waves with frequency ω, propagating at an angle $\theta = \arcsin(\omega/N)$ to the horizontal. The beam is excited by a point dipole in an exponentially stratified liquid. The stream function of the initial field in a system of coordinates (p, q) attached to the beam, has the form [9]

$$\psi(p,q) = D_0\int_0^\infty \exp(ikp - \delta_\nu^2 k^3 q)dk, \quad \delta_\nu = \left(\frac{\nu}{2N\cos\theta}\right)^{1/2} \quad (2.13)$$

The pattern of isolines of the stream function Ψ_s (they are tangential to the velocity field), calculated from (2.10) and (2.13), are shown in the figure in (p^*, q^*) coordinates, normalized to the scale of the internal boundary flow $\delta_\nu : p^* = p/\delta_\nu$, $q^* = q/\delta_\nu$. The arrows indicate the directions of steady motion of the liquid. Hence, the phase velocity and the Stokes drift velocity are in opposite directions at the centre of the beam of internal waves.

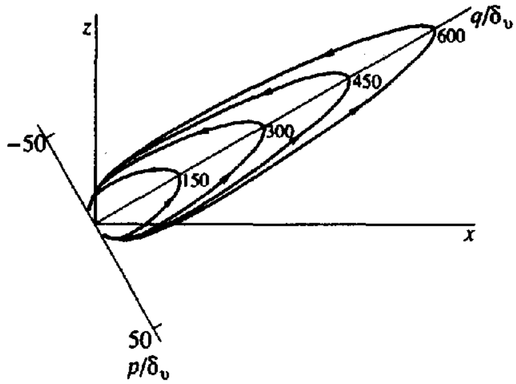

Fig. 1

3. THE STEADY FORCE OF INTERNAL WAVES

We will consider the steady component of the force of a beam of periodic internal waves, incident on a horizontal rigid surface $z = 0$. The stream function of the beam, generated by a compact source, can be represented in the form [9]

$$\psi(p,q) = \int_0^\infty D(k)\exp(ikp - \delta_\nu^2 k^3 q)dk \tag{3.1}$$

where $D(k)$ is the spectral function of the source and (p, q) is the attached system of coordinates (the q axis is directed along the beam). The total field, which consists of the incident and reflected beams, and also the internal boundary flow in the plane, has the form [9]

$$\psi(x,z) = \int_0^\infty A(k)\varphi(k,z)e^{ikx}dk \tag{3.2}$$

$$\varphi(k,z) = e^{ik_w z} - \frac{k_b + k_w}{k_b - k_w}e^{-ik_w z} + \frac{2k_w}{k_b - k_w}e^{-ik_b z} \tag{3.2}$$

$$k_w(k) = -k\operatorname{ctg}\theta + \frac{i\delta_\nu^2 k^3}{\sin^4\theta}, \quad k_b(k) \approx (1+i)\sqrt{\frac{\omega}{2\nu}}$$

where

$$A(k) = \frac{1}{\sin\theta}D\left(\frac{k}{\sin\theta}\right)\exp\left(-\frac{\delta_\nu^2 k^3 L}{\sin^3\theta}\right)$$

Here L is the distance from the source to the plane, measured along the beam.

It follows from the first relation of (3.2) that

$$\psi_x(x,0) = \psi_z(x,0) = 0 \tag{3.3}$$

The normal steady component $\mathcal{F}_z$ and the tangential steady component $\mathcal{F}_x$ of the total force acting on unit area of the reflecting plane, can be represented in the form [6]

$$\mathcal{F}_z = P_s(x,0) + 2\rho_0\nu\Psi_{sxz}\big|_{z=0}, \quad \mathcal{F}_x = \rho_0\nu(\Psi_{sxx} - \Psi_{szz})\big|_{z=0} \tag{3.4}$$

where, by virtue of relations (3.3), $\mathcal{F}_x = 0$. The pressure $P_s(x, 0)$ is found from the first equation of (2.9), which, taking (3.3) into account, can be written in the form

$$P_{sx}(x,0) = \rho_0\nu\Psi_{szzz}\big|_{z=0}$$

Substituting (2.10) and (3.2) here, we obtain

$$P_s(x,0)=\frac{3i\rho_0\nu}{4\omega}\int_0^\infty\int_0^\infty\frac{k'}{k'-k''}A(k')A^*(k'')\times$$

$$\times\ \varphi_{zz}(k',0)\varphi^*_{zz}(k'',0)e^{i(k'-k'')x}dk'dk''+\text{c.c.} \quad (3.5)$$

Integrating (3.5) with respect to x from $-\infty$ to $+\infty$, we obtain the force acting on an infinite strip of unit width (along the y axis),

$$\tilde{\mathcal{F}}=\frac{3i\pi\rho_0\nu}{2\omega}\int_0^\infty k[|A(k)|^2\,h'(k)h^*(k)+|h(k)|^2\,A'(k)A^*(k)]dk+\text{c.c.}$$

$$h(k)=2k_w(k)[k_b(k)+k_w(k)] \quad (3.6)$$

The prime denotes a derivative with respect to k.

If the beam is excited by a point dipole a distance L along the beam from the plane, we have

$$A(k)=\frac{D_0}{\sin\theta}\exp\left(-\frac{\delta_\nu^2k^3L}{\sin^3\theta}\right)$$

and, taking the expressions for k_w and k_b from (3.2) into account, we obtain

$$\tilde{\mathcal{F}}=-18\pi\rho_0\,v_m^2\,\frac{\cos^3\theta}{\sin\theta}\sqrt{\frac{\nu}{2\omega}}(\delta_\nu^2L)^{2/3}\left[\Gamma\left(\frac{2}{3}\right)\right]^{-1} \quad (3.7)$$

Here we have introduced the maximum oscillatory velocity in the beam v_m at the point where it is incident on the reflecting plane.

Hence, the steady component of the force generated by a beam of internal waves, incident on a horizontal reflecting plane, increases as the square of its amplitude and also increases monotonically as the frequency of the wave decreases (it follows from the form of (3.7) that in this case the source approaches the reflecting surface along the vertical and the area of the interaction spot increases). The expressions obtained are inapplicable as $\omega\to 0$, since the calculations of the roots of the dispersion equation are approximate. The use of the exact solutions of the dispersion equation removes the "infrared" divergence in (3.7).

The force acting on an inclined reflecting surface should be calculated in a similar way using the solutions of the dispersion equation of the internal waves in a local system of coordinates connected with them.

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